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Correlations and dynamics in ensembles of maps: Simple models

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Interesting correlations between individual maps have been reported in systems of globally coupled maps and of uncoupled maps submitted to a global noise. These have been described under the heading "violation of the law of large numbers." An elementary explanation of this phenomenon is proposed. It is illustrated in a simple approximation for logistic maps. We then introduce an alternative model of coupled homographic maps which is much easier to study. A precise analysis of the emergent collective dynamics is given.

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I. INTRODUCTION

Dynamics in systems with many degrees of freedom is an area of active investigations [1,2] in fields ranging from hydrodynamics to neural networks. Globally coupled systems form a particularly simple class [1,3–6] where interesting cooperative dynamical properties have been observed [3,7–11]. The present work is motivated by intriguing observations in ensembles of maps. Kaneko [3] has studied the dynamics of a system of logistic maps globally coupled through their mean:

$$x_{t+1}(i)=(1-\epsilon)f(x_t(i),a)+\epsilon h_t \,, \tag{1}$$

$$h_{t,N}=\frac{1}{N}\sum_{i=1}^{N} f(x_t(i),a) \,, \tag{2}$$

where $\{t\}$ is a discrete time index, $\{i\}$ is the map index which varies from 1 to N, and ϵ the coupling parameter. $f(x,a)$ is the logistic map

$$f(x,a)=1-ax^2 \,. \tag{3}$$

The value of a is chosen close to 2 so that the individual maps are fully chaotic. In the uncoupled case ($\epsilon=0$), the N variables behave as independent random variables. So, the fluctuations of $h_{t,N}$ decrease like $1/\sqrt{N}$ and vanish in the limit of a large number of maps ($N\rightarrow\infty$). On the other hand, for any small positive coupling ($\epsilon>0$), it is found that the mean field $h_{t,N}$ does not relax to a constant value and has a nonzero variance even in the limit of infinite system size. This is the so-called "violation of the law of large numbers" [3]. An interesting simplifying step is taken in [12] where the dynamics of an ensemble of uncoupled logistic maps submitted to a common noise term is numerically studied. The main observation of [12] is that in this case a "violation of the law of large numbers" is observed as well (i.e., the mean

$$h_{t,N}=(1/N)\sum_{j=1}^{N} f(x_t(j))$$

fluctuates in time, even in the large N limit). As the name implies, in both cases the phenomenon has been attributed to an anomalous dependence of the fluctuations on the system size. In the following sections we argue instead that the phenomenon arises because the sharply defined infinite size limit mean field $h_t(=\lim_{N\rightarrow\infty}h_{t,N})$ has a nontrivial dynamics. Therefore, h_t varies in time around its time-averaged mean value and the histogram of its value has a nonzero width. In Sec. II we consider the simplest case of uncoupled maps. We argue that the h_t dynamics is created by correlations between different maps created by the common noise forcing term though it is random in time. This is illustrated by an elementary example. The logistic map invariant measures are very complicated [13] as well as the relaxation toward them. So, the case of uncoupled logistic maps submitted to a global noise is treated in a simple approximation. Nonetheless, this gives results in reasonable agreement with the numerics of [12]. In Sec. III we discuss the more interesting case of globally coupled maps. For logistic maps, the existence of a nontrivial dynamics of h_t can be seen in the power spectrum of the time series of h_t where broad peaks emerge [3] or in return maps where h_{t+1} is plotted as a function of h_t [10]. In order to show simply how correlations between individual maps induced by a common forcing term can result in a nontrivial dynamics of the mean field when the maps are globally coupled, we introduce a new model of coupled homographic maps. In this latter case, the invariant measures are fully known as well as the relaxation toward them. This allows us to show explicitly the emergence of a cooperative dynamics and to analyze precisely the mean field motion.

II. CORRELATIONS IN AN UNCOUPLED ENSEMBLE OF CHAOTIC MAPS SUBMITTED TO A SPATIALLY UNIFORM NOISE

We begin by considering correlations in uncoupled ensembles of chaotic maps submitted to a common noise term. In [12] the following ensemble of logistic map is studied:

$$x_{t+1}(i)=1-a_t[x_t(i)]^2 \,. \tag{4}$$

a_t is chosen to fluctuate temporally around its mean value a but is the same for every map at any given time. The numerical observation of [12] is that in this case also the mean field h_t [Eq. (2)] fluctuates in the large N limit although the individual maps are fully chaotic and uncoupled ($\epsilon=0$). As shown in Fig. 1, the mean square deviation (MSD) of h, Δ_N, decreases to a small but finite value as $N\to\infty$:

$$\lim_{N\to\infty}\Delta_N>0 \quad \text{with } \Delta_N\equiv\overline{h_{t,N}^2}-\overline{h_{t,N}}^2 , \tag{5}$$

where the overbar denotes time average.

A simple example shows why this phenomenon is expected. Instead of being generated by iterating the logistic map $f(x,a)$, let us suppose that the sequence of $x_t(i)$ is produced by a Bernoulli shift with two symbols $1,-1$ (which could be thought of as coming from a symbolic dynamics description of the logistic map) so that the value 1 is obtained with probability $p(a)$ and -1 with probability $q(a)$ $[=1-p(a)]$ depending on an external parameter a. For fixed a, it is clear that the mean field $(1/N)\sum_{j=1}^{N}x_t(j)$ has the value $p(a)-q(a)$ and that it does not fluctuate in the large N limit. Now, let us consider the case where a is a random variable a_t as in Eq. (4). Then, each individual $x_t(i)$ has a probability $p(a_t)$ of having the value $+1$ [and the probability $q(a_t)=1-p(a_t)$ of having the value -1]. So, each individual map is a simple Bernoulli shift with probability $p=\overline{p(a)}$ and $q=\overline{q(a)}$ where the mean value (denoted by an overbar) is taken over the probability distribution of a. Nonetheless, in the large N limit, the mean field of an ensemble of such maps has the sequence of values $p(a_t)-q(a_t)$ and it fluctuates as much as a_t. This is the "violation of the law of large numbers." In this simple context, it is clearly seen that the phenomenon appears because different maps are correlated since their individual probability distribution at a given time (i.e., the invariant probability distribution of the chaotic map) depends on the value of the common noise term. What makes this example very easy to deal with is that (i) the probability distribution is known for each value of the external parameter and (ii) the relaxation towards the new invariant distribution when the external parameter is changed is particularly simple, being complete in one time step.

Although we believe that the explanation of the results of [12] is basically analogous, a complete analysis is difficult to perform because these two features are no longer true for the logistic map in the chaotic regime away from the point $a=2$. So, we simply present here an approximate calculation to support our view. We assume that the set of maps relax in one time step to the new invariant measure when the parameter a is changed (an "adiabatic" approximation). So, as the parameter a is changed with time, the system wanders from one invariant measure to the other. This produces fluctuations in h_t exactly as in the above simple example, since the value of the mean field depends on the invariant distribution used to calculate it.

Let us detail this reasoning. We first consider the case where a is fixed. Support that a large number N of initial points $x_0(1),\dots,x_0(N)$ are picked at random in $[-1,1]$. After t iterations with the logistic map $f(x,a)$, N points $x_t(1),\dots,x_t(N)$ are obtained which in the limit $N\to\infty$ are distributed according to the density function $\rho_t(x,a)$ [14]. This density relaxes toward the invariant density $\rho(x,a)$ as $t\to\infty$. Correspondingly, in the limit of an infinite number of maps the mean field h_t [Eq. (2)] is simply the average of $f(x,a)$ over the density $\rho_t(x,a)$. As $t\to\infty$, it relaxes toward the first moment $h(a)$ of $\rho(x,a)$:

$$h(a)=\lim_{N\to\infty}h_N=\int dx\,\rho(a,x)f(x)=\int dx\,\rho(a,x)x . \tag{6}$$

Of course, for a finite N, $h_{t,N}$ fluctuates around $h(a)$. Since one is summing N random uncorrelated variables distributed according to the probability distribution $\rho(x,a)$, the fluctuations of $h_{t,N}(a)$ for t large are given by

$$\langle[h_{t,N}(a)]^2\rangle-[h(a)]^2=\frac{1}{N}\left\{\int dx\,\rho(a,x)[x-h(a)]^2\right\} , \tag{7}$$

where the average is taken over different initial conditions of N maps. So the fluctuations of $h_{t,N}$ die out like $1/N$ when the number of maps is increased ("the law of large numbers").

Now, let us see how the picture changes when a itself fluctuates in time. Then, the fluctuations of $h_{t,N}(a_t)$ come from two sources: the finite number of maps and the fact that the invariant measure and therefore $h(a)$ depends on the value of a.

In order to obtain a simple estimate of the relative magnitude of the two effects, we make the assumption that the relaxation toward the asymptotic density $\rho(x,a)$ is complete in one time step [i.e., $\rho_{t=1}(x,a)=\rho(x,a)$]. So, as above we obtain

$$\langle[h_{t,N}(a_t)]^2\rangle=[h(a_t)]^2+\frac{1}{N}\left\{\int dx\,\rho(a_t,x)[x-h(a_t)]^2\right\} , \tag{8}$$

where again the average is taken over different initial conditions of N maps.

The fluctuations around the time-averaged mean field are therefore obtained by averaging Eq. (8) over time. The time average can be replaced by an average over the distribution of a (which we also denote by an overbar). Finally, we obtain for the mean field mean square fluctuations Δ_N [Eq. (5)]:

$$\Delta_N=\overline{[h(a)]^2}-\overline{[h(a)]}^2+\frac{1}{N}\overline{\int dx\,\rho(a,x)[x-h(a)]^2} . \tag{9}$$

Therefore, when the number of maps is increased, the fluctuations in time of $h_{t,N}$ do not die out ("the violation of the law of large numbers"). We can compare the simple estimate of Eq. (9) with the numerical simulation results of [12]. There, a_t was chosen as

$$a_t=a(1+\sigma\eta_t) , \tag{10}$$

where σ is the magnitude of the fluctuation and η_t is a random number uniformly distributed in $[-0.5,0.5]$. The estimated time average mean field $\bar{h}$ and fluctuations δh^2 [Eq. (9)] are

$$\bar{h}=\overline{h(a)}=\frac{1}{a\sigma}\int_{a(1-\sigma/2)}^{a(1+\sigma/2)} da\, h(a) , \tag{11}$$

$$\delta h^2=\overline{[h(a)]^2}-[\overline{h(a)}]^2 = \frac{1}{a\sigma}\int_{a(1-\sigma/2)}^{a(1+\sigma/2)} da\,[h(a)-\bar{h}]^2 . \tag{12}$$

These expressions have been estimated by sampling the function $h(a)$. For each value of a, the invariant distribution was estimated by iterating 800 times 10^4 initial random points. Only the last 400 iterations were taken into account in order to allow the relaxation of transients. For $\sigma=0.01$, $a=1.98$, we obtained

$$\bar{h}=(8.2\pm0.2)\times10^{-2} ,$$

$$\delta h^2=(5.0\pm0.5)\times10^{-4} ,$$

while a full simulation (along the lines of [12]) gives $\bar{h}=8.8\times10^{-2}$, $\delta h^2=4.3\times10^{-4}$. For $\sigma=0.02$, $a=1.98$, comparable agreement is obtained. The adiabatic approximation gives

$$\bar{h}=(7.4\pm0.4)\times10^{-2} ,$$

$$\delta h^2=(9\pm1)\times10^{-4} ,$$

and the full simulation $\bar{h}=8.0\times10^{-2}$, $\delta h^2=8.3\times10^{-4}$. In order to compare the full N dependence, the coefficient of $1/N$ in Eq. (9) should be estimated. It is equal to $\frac{1}{2}$ for $a=2$ where the invariant distribution is explicitly known. A close result of about 0.46 is obtained in the above two cases ($a=1.98$, $\sigma=0.01$ or 0.2). The adiabatic prediction is compared with the results of a full simulation in Fig. 1. The agreement is quite good and in fact better than could have been expected from our rather crude approximation. This leads us to think that the essence of the phenomenon has been captured.

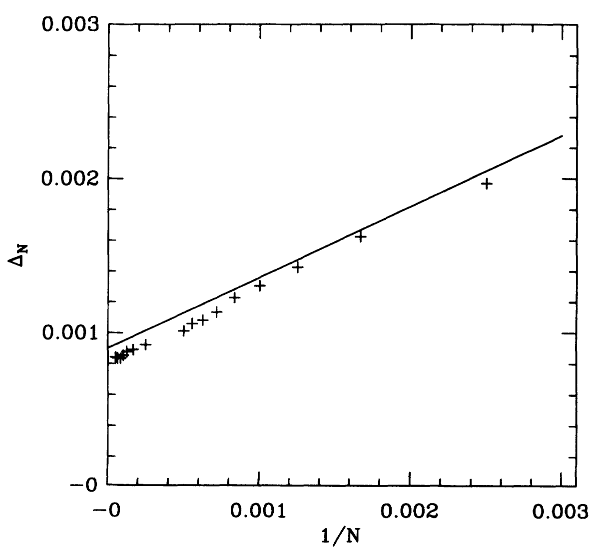

FIG. 1. Mean square deviation Δ_N of the mean field vs the number of maps N from a direct simulation of the set of maps (plus signs) [Eqs. (4) and (10)] and in the adiabatic approximation (solid line) [Eq. (9)]. The parameters are $a=1.98$, $\sigma=0.02$.

III. ENSEMBLES OF HOMOGRAPHIC MAPS

In Sec. II it has been shown that when a global term is forcing an ensemble of maps, the individual maps values $x_t(1),\ldots,x_t(N)$ become correlated because the invariant density of each map depends on the global forcing. When the forcing term is not imposed externally, as was the case in Sec. II, but itself depends on the values of the $x_t(j)$ a nontrivial collective dynamics can result. At the level of the previous adiabatic approximation, the relation $h_{\text{out}}=g(h_{\text{in}})$ can be numerically obtained. In this framework the emergence of a collective dynamics amounts to finding whether this iteration reaches a fixed point [10]. This approach has two obvious weak points. Firstly, stability is tested only for a very restricted set of perturbation so it can be strongly overestimated. Moreover, for the logistic map, g is a very irregular function and the adiabatic approximation clearly needs some refinements to be compared meaningfully with the simulations [3] of the full system. Instead of pursuing this difficult strategy, we follow an easier path. We choose a map for which the invariant density is known for all parameter values and for which the relaxation toward the invariant density is easily studied. The corresponding globally coupled system can then be analyzed and the collective behavior analytically characterized [15].

A. Properties of a single map

Motivated by the solvable Lloyd model [16] of a quantum particle in a random potential, we consider the homographic mapping

$$x_{t+1}=a-b/x_t , \quad a>0 , \; b>0 . \tag{13}$$

In the following we consider the case $a^2-4b<0$ when this mapping has no real fixed point and the distribution of successive iterates $x_0,x_1,\ldots,x_t,\ldots$ is nontrivial. It can be very explicitly obtained by expressing x_t in terms of x_0 (using the well-known fact that the composition of homographic maps is equivalent to the product of associated 2×2 matrices; see the Appendix):

$$x_t=\sqrt{b}\,[\cos(\theta^*)-\sin(\theta^*)\tan(t\theta^*+\phi)] , \tag{14}$$

where

$$\theta^*=\arctan(\sqrt{4b/a^2-1})$$

and ϕ depends on the value of the initial condition x_0 $\{\tan(\phi)=\cot(\theta^*)=x_0/[\sqrt{b}\sin(\theta^*)]\}$. When θ^* is not commensurate with π, $t\theta^*$ [mod(π)] is uniformly distributed in $[-\pi/2,\pi/2]$ and the distribution of x_t in the limit $n\to\infty$ converges toward the Cauchy distribution:

$$\rho_{r^*,s^*}(y)=\frac{1}{\pi}\frac{s^*}{(x-r^*)^2+s^{*2}} , \tag{15}$$

with

$$r^*=a/2 , \quad s^*=\tfrac{1}{2}\sqrt{4b-a^2} . \tag{16}$$

A point of view which is complementary to this average over a trajectory and more relevant for the behavior of an assembly of maps is to consider the evolution of densities [14]. Consider a set of points which are distributed according to a Cauchy distribution with parameters r_t, s_t [Eq. (15)]. The new distribution after one iteration is given by

$$\rho_{r_{t+1},s_{t+1}}(y)=\int dx\, \rho_{r_t,s_t}(x)\delta(y-(a-b/x)) . \quad (17)$$

The new distribution of maps is again a Cauchy distribution with new parameters r_{t+1}, s_{t+1} which are simple functions of r_t, s_t. Defining the complex numbers $z_t = r_t + is_t$, a direct calculation shows that the relation between (r_t, s_t) and (r_{t+1}, s_{t+1}) can be expressed as

$$z_{t+1}=a-b/z_t . \quad (18)$$

The simplicity of the map given by Eq. (13) is that the evolution of a density is reduced to the evolution of a complex number.

It can be wondered if densities tend to the invariant Cauchy density [Eq. (15)] under iteration by the homographic mapping [Eq. (13)]. In other terms, is the fixed point $x^* = r^* + is^*$ (16) an attractive fixed point for the complex mapping (18)? The answer which may appear surprising at first sight is that it is not the case. First, by linearization the fixed point, z^* is found to be marginally stable for the homographic mapping [Eq. (18)]. In order to elucidate fully the mapping dynamics in the complex plane, it is convenient to introduce new complex coordinates (see the Appendix). We define a variable w in the following way:

$$z=\sqrt{b}\,\frac{w+e^{i\theta^*}}{e^{i\theta^*}w+1} . \quad (19)$$

It is straightforward to show that as z transforms according to Eq. (18), w obeys

$$w_{t+1}=e^{-2i\theta^*}w_t , \quad (20)$$

which means that, for generic θ^*, w draws a circle. Since the transformation from z to w is homographic, z draws a circle on the complex plane as well. The specific circle depends, of course, on the initial condition. Therefore, an arbitrary Cauchy distribution does not relax to the invariant distribution (15). The underlying reason is that the homographic mapping [Eq. (13)] is ergodic but not mixing so that the initial correlation between two points does not decay with time [as shown by the explicit formula Eq. (14)]. In order to remove this undesirable feature, we slightly modify our model by adding independent random noise to each map

$$x_{t+1}(j)=a-b/x_t(j)+\xi_t(j) . \quad (21)$$

The simplicity of density evolution is preserved [16] if we choose the noise probability distribution to be a Cauchy distribution $\rho_{0,\eta}(\xi)$ since a linear combination of Cauchy variable is also a Cauchy variable. Starting from a distribution of the maps $\rho_{r_t,s_t}(x)$, the distribution after one iteration is given by

$$\rho_{r_{t+1},s_{t+1}}(y) =\int dx\, d\xi\, \rho_{0,\eta}(\xi)\rho_{r_t,s_t}(x)\delta(y-(a-b/x+\xi)) . \quad (22)$$

$\rho_{r_{t+1},s_{t+1}}(x)$ is a Cauchy distribution as before but, instead of Eq. (18), one has

$$z_{t+1}=a-b/z_t+i\eta . \quad (23)$$

Linearization around the fixed point of this equation now shows that for any small η the fixed point is stable. It coincides with $z^*=r^*+is^*$ [Eq. (16)] as $\eta \to 0$. The noise destroys the correlations among maps, and densities relax toward the invariant Cauchy distribution [Eq. (15)] as iteration proceeds.

B. Globally coupled maps

Having analyzed the dynamics of a single homographic map [Eqs. (13) and (21)], we can study the behavior of a globally coupled system of such maps. That is, we study Eq. (1) with $f(x)$ given by Eq. (21).

Before proceeding further, one point should be clarified. One might think that if at time t the maps are distributed according to $\rho_t(x)$, when the mean field h_t is determined as

$$h_{t,N}=\frac{1}{N}\sum_{j=1}^{N} f(x_t(j)) \xrightarrow[N\to\infty]{} h_t=\int dx\, \rho_t(x)f(x) . \quad (24)$$

In our case, this gives

$$h_t=\int dx\, d\xi(a-b/x+\xi)\rho_{r_t,s_t}(x)\rho_{0,\eta}(\xi) =\mathrm{Re}(a-b/z_t) . \quad (25)$$

Equation (24) is true without further qualification for usual bounded maps for which $h_{t,N}$ is a well defined function of t in the limit $N\to\infty$. However, this is not so if the value $f[x_t(j)]$ are distributed according to a Cauchy distribution because the second moment of the distribution diverges. In this case, averaging does not suppress fluctuations. In fact, the average of independent Cauchy variables is a Cauchy variable with the same distribution. It could be interesting to study the evolution of a globally coupled system of homographic maps taking this feature into account. The mean field h_t would then be a stochastic Cauchy variable distributed according to the density at time t, while the complex parameter z_t characterizing the density would follow a stochastic recursion relation depending on h_t. We choose instead to stay closer to the usual case by modifying the mean-field definition so as to retain the result of Eq. (25). Since fluctuations of h_t come from the dominating effect of rare events for which $f(x_t(j))$ is extremely large, we introduce a saturation function S which bounds the influence of these rare events:

$$h_{t,N}=\frac{1}{N}\sum_{j=1}^{N} S(f(x_t(j))) , \quad (26)$$

where $S(y)=y$ for $|y|<A$ and $S(y)=A$ otherwise. With

this definition, h_t is well defined in the limit $N\to\infty$ and it can be evaluated by an integration over the density as in Eq. (24). In the large A limit, the result (25) is retained.

So, the dynamics of a system of globally coupled homographic maps with a coupling parameter ϵ is chosen to be given by

$$x_{t+1}(j)=(1-\epsilon)[a-b/x_t(j)+\xi_t(j)]+\epsilon h_t\ , \tag{27}$$

where h_t is given by Eq. (26) with some very high cutoff A. With this dynamics, a Cauchy distribution $\rho_{r_t,s_t}(x)$ is evolving into another Cauchy distribution $\rho_{r_{t+1},s_{t+1}}(x)$ in the following way:

$$z_{t+1}=(1-\epsilon)(a-b/z_t+i\eta)+\epsilon\,\mathrm{Re}(a-b/z_t)\ , \tag{28}$$

where as before $z_t=r_t+is_t$. Again, the simplicity of the model is that the evolution of the full distribution of coupled maps is reduced to the simple iteration (28). So, questions about the invariant distribution, its stability, and the emergence of a nontrivial dynamics can be easily addressed.

1. The invariant distribution

The invariant distribution is the Cauchy distribution which corresponds to a fixed point of Eq. (28). Looking for it under the form $z_p=Re^{i\theta}$, one obtains

$$\begin{aligned}&(R^2+b)\cos(\theta)=aR\ ,\\&[R^2-b(1-\epsilon)]\sin(\theta)=\eta R(1-\epsilon)\ .\end{aligned} \tag{29}$$

In the small noise limit ($\eta\ll 1$), which is the only one of concern for us here, these equations can be easily solved perturbatively. One obtains

$$\begin{aligned}&R=\sqrt{b(1-\epsilon)}+\eta\frac{1-\epsilon}{2\sin(\theta_\epsilon)}+O(\eta^2)\ ,\\&\theta=\theta_\epsilon-\eta\frac{a\epsilon(1-\epsilon)}{8b(1-\epsilon/2)^2\sin^2(\theta_\epsilon)}+O(\eta^2)\ ,\end{aligned} \tag{30}$$

with

$$\cos(\theta_\epsilon)=\frac{a(1-\epsilon)^{1/2}}{2\sqrt{b}\,(1-\epsilon/2)}\ ,\quad 0<\theta_\epsilon<\pi/2\ .$$

2. Stability of the invariant distribution

The stability of this invariant distribution under the dynamics of Eq. (27) depends on the competition between the noise (η), which tends to stabilize it, and the coupling (ϵ) which tends to synchronize the different maps. This can be precisely and simply quantified in the two-parameter space of possible Cauchy distributions by linearizing the recursion relation (28) around the fixed point (30) as $z_t=z_{\mathrm{FP}}+\delta z_t$. One obtains

$$\delta z_{t+1}=\frac{b(1-\epsilon/2)}{z_{\mathrm{FP}}^2}\delta z_t+\frac{b\epsilon}{\overline{z_{\mathrm{FP}}}^2}\overline{\delta z_t}\ , \tag{31}$$

where the overbar denotes complex conjugation. Separating real and imaginary parts as $\delta z_t=u_t+iv_t$ gives

$$\begin{aligned}&u_{t+1}=\frac{b}{R^2}[\cos(2\theta)u_t+\sin(2\theta)v_t]\ ,\\&v_{t+1}=\frac{b(1-\epsilon)}{R^2}[-\sin(2\theta)u_t+\cos(2\theta)v_t]\ .\end{aligned} \tag{32}$$

To linear order in ϵ and η, the eigenvalues of this linear recursion relation are

$$\lambda_\pm=e^{\pm 2i\theta}\left[1+\left[\frac{\epsilon}{2}-\frac{\eta}{\sqrt{b}\,\sin(\theta_\epsilon)}\right]+O(\epsilon^2,\epsilon\eta,\eta^2)\right]\ . \tag{33}$$

The invariant distribution is destabilized when the modulus of these eigenvalues becomes greater than 1. So, a nontrivial dynamics appears as soon as

$$\epsilon>\epsilon_c=\frac{2\eta}{\sqrt{b-a^2/4}}\ . \tag{34}$$

This exact criterion can be compared to the prediction of an adiabatic approximation. In this case Eq. (28) would be replaced by

$$z_{t+1}=(1-\epsilon)(a-b/z_t+i\eta)+\epsilon h_{\mathrm{in}}\ . \tag{35}$$

The invariant distribution corresponds to the fixed point of this recursion relation which is in the small η limit $z_{\mathrm{ad}}=\sqrt{b(1-\epsilon)}e^{i\theta_{\mathrm{ad}}}$, with

$$\cos(\theta_{\mathrm{ad}})=[(1-\epsilon)a+\epsilon h_{\mathrm{in}}]/2\sqrt{b(1\epsilon)}\ .$$

It can be used to compute h_{out} as

$$h_{\mathrm{out}}=\mathrm{Re}(a-b/z_{\mathrm{ad}})\ . \tag{36}$$

Imposing the self-consistency relation $h_{\mathrm{in}}=h_{\mathrm{out}}$ gives back $\theta_{ad}=\theta_\epsilon$ so that the fixed point in the adiabatic approximation coincides with the fixed point of the complete iteration ($z_{\mathrm{ad}}=z_{\mathrm{FP}}$) as it should. However, Eq. (36) gives as a sufficient condition for stability of the invariant distribution:

$$\frac{\epsilon}{2(1-\epsilon)}<1\ .$$

This is clearly much weaker than the criterion (34). So, in this case, the stability of the invariant distribution is strongly overestimated in the adiabatic approximation.

3. Analysis of the collective dynamics

It is interesting to characterize more completely the emergent collective dynamics when the invariant distribution is unstable. For ϵ and η small and of comparable magnitude, the usual techniques of weakly nonlinear analysis can be applied [17]. We have found it convenient to use the w coordinates [Eq. (19)] in which the unperturbed motion is a simple rotation. It is then simple to use averaging and obtain the mean displacement due to the perturbation (see the Appendix):

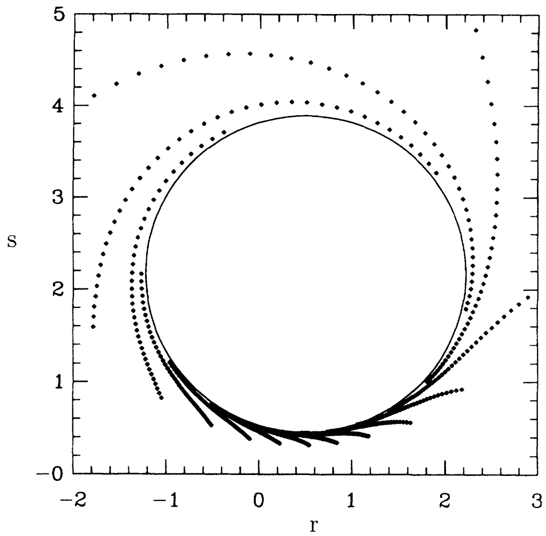

FIG. 2. Comparison between the trajectories in the (r,s) plane obtained by iterating Eq. (28) (diamonds) and the limit circle predicted by Eq. (37) (solid line). The parameters are $a=1$, $b=2$, $\eta=0.01$, $\epsilon=0.02$ ($\epsilon_c=1.512\times10^{-2}$).

$$\langle w_{t+1}-e^{-2i\theta^*}w_t\rangle = \tfrac{1}{2}e^{-2i\theta^*}w_t\left\{\left[\epsilon-\frac{2\eta}{\sqrt{b}\,\sin(\theta^*)}\right] -\epsilon|w_t|^2+O(\epsilon^2,\eta\epsilon,\eta^2)\right\}. \quad (37)$$

So, in the w coordinates the dynamics created by the fixed point instability is a simple rotation around a circle of radius

$$R_c=[1-2\eta/\epsilon\sqrt{b}\,\sin(\theta^*)]^{1/2}$$

[18]. When reverting to the z coordinates, it is found that the Cauchy distribution $\rho_{r,s}(x)$ follows a circle in the (r,s) plane of center $(\cos(\theta^*),\ \sqrt{b}[(1+R^2)/(1-R^2)]\sin(\theta^*))$ and radius

$$\sqrt{b}\,[2R_c/(1-R_c^2)]\sin(\theta^*)\ .$$

It is interesting to note that the truncated expansion (37) is accurate for $\epsilon\sim\eta$ and therefore it consistently describes circles of radius of order 1. The more usual $\sqrt{\epsilon}$ magnitude is not obtained because restabilizing nonlinearities only appear through the coupling term and are therefore of order ϵ (and not as usual of order 1). As shown in Fig. 2, the prediction of Eq. (37) compares well with a direct simulation of the iteration (28).

IV. CONCLUSION

We have analyzed correlations between different maps and their consequences in globally coupled systems using simple models. We have first tried to clarify the phenomenon described as a "violation of the law of large numbers." It has been argued that this is coming from time fluctuations of the mean field arising from correlations between different maps induced by the common forcing term which modifies the map invariant distribution. For uncoupled systems of maps submitted to a global noise, the magnitude of the correlations has been found to be correctly estimated by a simple adiabatic approximation. In order to see further how the correlations between different maps can induce a nontrivial collective dynamics, we have introduced and analyzed a globally coupled model of homographic maps. A precise criterion for the emergence of a collective dynamics has been obtained and this global motion has been characterized.

Several questions remain for further investigations. We expect that a collective dynamics appears in a large class of globally coupled systems but finding a simple criterion would be nice. A precise analysis of the emergent dynamics for globally coupled logistic maps also seems a challenging task. On a more qualitative level, an understanding of the peculiar influence of noise in this [3,10] and other [8,9] globally coupled systems would also be very interesting. Finally, physical realizations of these systems would be quite welcome.

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APPENDIX

Computations with homographic maps are conveniently done by using their relation with 2×2 matrices. To each homographic map $m(z)$, one can associate a 2×2 Matrix $\mathbf{M}$:

$$m(z)=\frac{az+b}{cz+d}\rightarrow\mathbf{M}=\begin{bmatrix}a & b\\ c & d\end{bmatrix}. \quad (A1)$$

Under this correspondence, composition of homographic maps is equivalent to matrix multiplication. One can directly obtain $m(z)$ from the matrix M by evaluating the product of M with the vector $\binom{z}{1}$ and by taking the ratio of the upper and lower components of the resulting vector.

So, to study the iteration

$$z_{t+1}=l(z_t)=a-b/z_t\ ,\quad a>0\ ,\ b>0\ ,\ a^2-4b<0\ , \quad (A2)$$

it is useful to introduce the matrix $\mathbf{L}$:

$$\mathbf{L}=\begin{bmatrix}a/\sqrt{b} & -\sqrt{b}\\ 1/\sqrt{b} & 0\end{bmatrix}=\mathbf{PDP}^{-1}\ , \quad (A3)$$

where the diagonalization of $\mathbf{L}$ involves the two matrices

$$\mathbf{P}=\begin{bmatrix}\sqrt{b} & \sqrt{b}\,e^{i\theta^*}\\ e^{i\theta^*} & 1\end{bmatrix}\ ,\quad \mathbf{D}=\begin{bmatrix}e^{-i\theta^*} & 0\\ 0 & e^{i\theta^*}\end{bmatrix}\ , \quad (A4)$$

The decomposition (A3) can be interpreted as a composition of homographic transformations $l=p\circ d\circ p^{-1}$. So, in the w coordinates such that $z=p(w)$, l is reduced to the homographic transformation d associated to the diagonal matrix **D**, i.e., l is a simple rotation of angle $-2\theta^*$. The expression of $p(w)$ given in the main text [Eq. (19)] is readily obtained from the expression of the associated matrix **P** given above [Eq. (A4)]:

$$z=p(w)=\sqrt{b}\,\frac{w+e^{i\theta^*}}{e^{i\theta^*}w+1}\ . \tag{A5}$$

x_t can easily be expressed as a function of x_0 by using a decomposition over the eigenvectors of **L**:

$$\begin{bmatrix} x_0 \\ 1 \end{bmatrix}=\frac{1}{2\cos(\phi)}\left\{e^{i\phi}\begin{bmatrix}\sqrt{b}\,e^{i\theta^*}\\ 1\end{bmatrix}+e^{-i\phi}\begin{bmatrix}\sqrt{b}\,e^{-i\theta^*}\\ 2\end{bmatrix}\right\}, \tag{A6}$$

with

$$x_0=\sqrt{b}\,\cos(\theta^*+\phi)/\cos(\phi)$$

or, equivalently,

$$\tan(\phi)=\cos(\theta^*)-x_0/[\sqrt{b}\,\sin(\theta^*)]\ .$$

Therefore, the action of $\mathbf{L}^t$ is given by

$$\mathbf{L}^t\begin{bmatrix} x_0 \\ 1 \end{bmatrix}=\frac{1}{2\cos(\phi)}\left\{e^{i\phi}\begin{bmatrix}\sqrt{b}\,e^{i\theta^*}\\ 1\end{bmatrix}e^{it\theta^*} +e^{-i\phi}\begin{bmatrix}\sqrt{b}\,e^{-i\theta^*}\\ 1\end{bmatrix}e^{-it\theta^*}\right\}. \tag{A7}$$

x_t is simply given by the ratio of the upper and lower component of Eq. (A7):

$$\begin{aligned} x_t&=\sqrt{b}\,\frac{\cos[\phi+(t+1)\theta^*]}{\cos(\phi+t\theta^*)}\\ &=\sqrt{b}\,[\cos(\theta^*)-\sin(\theta^*)\tan(\phi+t\theta^*)]\ , \end{aligned} \tag{A8}$$

which is Eq. (14) of the main text.

The w coordinates are quite convenient for studying the dynamics of the globally coupled maps. Equation (28) reads in these coordinates

$$w_{t+1}=e^{-2i\theta^*}w_t+E(w_t)+O(\epsilon^2,\epsilon\eta,\eta^2)\ , \tag{A9}$$

with

$$E(w)=\frac{1}{p'(e^{2i\theta^*}w)}\left\{i\eta+\frac{\epsilon}{2}\left[\frac{1}{p(w)}-\frac{1}{\overline{p(w)}}\right]\right\}, \tag{A10}$$

where complex conjugation is denoted by an overbar and differentiation by a prime. The effect of the perturbation $E(w)$ can be estimated by averaging it along the unperturbed orbit which covers densely and uniformly a circle for θ^* incommensurate with π. Defining $v_t=\exp(2i\theta^* t)w_t$, Eq. (A9) is rewritten as

$$v_{t+1}=v_t+e^{2i\theta^*(t+1)}E(v_t e^{-2i\theta^* t})+O(\epsilon^2,\epsilon\eta,\eta^2)\ . \tag{A11}$$

Averaging the perturbation along the unperturbed orbit gives

$$\begin{aligned}\overline{E}(v)&\equiv\langle e^{2i\theta^*(t+1)}E(ve^{-2i\theta^* t})\rangle_t\\ &=e^{2i\theta^*}\int_0^{2\pi}\frac{d\phi}{2\pi}e^{i\phi}E(ve^{-i\phi})\ .\end{aligned} \tag{A12}$$

Using Eq. (A10), one obtains

$$\overline{E}(v)=\left[\frac{\epsilon}{2}-\frac{\eta}{\sqrt{b}\,\sin(\theta^*)}\right]v-\frac{\epsilon}{2}|v|^2v\ , \tag{A13}$$

which is Eq. (37) of the main text.

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[15] Another simple possibility is to generalize the Bernoulli shift approach of the previous section to globally coupled systems. As pointed out to us by A. S. Mikhailov, this is pursued in O. A. Druzhinin and A. S. Mikhailov, Radiofisika **32**, 444 (1989).

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[18] Note that θ, θ_ϵ, and θ^* coincide at linear order in the small parameters so that no rotation frequency shift appears in Eq. (37).